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Gauge-invariant anyon operators and spin-statistics relation in Chern–Simons matter field theory

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Abstract. We consider one type of gauge-invariant anyon operators from the complex scalar field theory minimally coupled to the Chern–Simons term in $2+1$ dimensions. In the Coulomb gauge condition, the anyonicity of operators, the multi-valued states, and the spin-statistics relation are derived by considering the proper distribution function, which appears in the definition of anyon operators. It is shown that the anyonicity is not a gauge artifact by also obtaining it in the covariant gauge.

1. Introduction

Fractional spin and statistics are particular features only in two spatial dimensions and the excitations exhibiting them are called anyons [1]. They have attracted much attention due to their possibile relevance to condensed matter phenomena, especially to the fractional quantum Hall effect [2] and high-T_c superconductivity [3]. Theoretical understanding of them has been gained in the context of both quantum mechanics and quantum field theory [4]. So far, development in the direction of field theory has not progressed as far as that of quantum mechanics. In the study of anyons at the field-theoretical level, the Abelian Chern–Simons (CS) theory minimally coupled to the matter fields is usually considered as the base system, and the existence of anyonic excitations is investigated as the first task. Such a task may be related to the problem of constructing the composite operator, which exhibits the anyonic properties, from the fundamental field variables of the base system. If the composite operator is properly constructed, it becomes the so-called anyon field operator.

In the canonical Hamiltonian formalism, anyon states are created by anyon operators acting on the vacuum state and should be physical such that the results obtained based on them are physically meaningful. This implies that the anyon operators should also be physical. In addition to this, the gauge invariance of the anyon operator has been emphasized [5].

Recently, one type of gauge-invariant anyon operator has been suggested and investigated based on the Maxwell–Chern–Simons theory in the axiomatic [6] and canonical [7] approach. The properties of anyon operators have been obtained in the covariant gauge condition. Because of its gauge invariance, it is expected that the same results could also be obtained in other gauges, especially in the Coulomb gauge. However, it does not seem so obvious since, for example, the algebraic and constraint structures of

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the CS theory in the Coulomb gauge are considerably different from those in the covariant gauge. Therefore, in order to give more concrete argument on the anyon operator, it is necessary to establish whether the properties of it are gauge independent. In the matter-coupled CS theory, the difference between the covariant and the Coulomb gauge condition is distinct. The features of the covariant gauge are that the gauge field, while it commutes with the matter fields, does not commute with itself, due to the symplectic structure, and the Gauss-law constraint is obeyed not strongly but weakly; that is, it leads to zero acting on the physical state. Here we note that, in the covariant gauge, the Gauss-law constraint is realized by the generator of Becchi–Rouet–Stora–Tyutin (BRST) symmetry, the BRST charge [8]. In the Coulomb gauge, a typical non-covariant gauge, the situation is completely reversed; the gauge field does no longer commute with the matter fields while it commutes with itself, and the Gauss law can be imposed strongly to zero, as will be shown in the later section.

In this paper, we consider the composite operators whose form is that suggested in [6, 7] for the study on the gauge-invariant anyon operators and take, as the base system, the complex scalar field theory minimally coupled to the CS term, which is given by

$$\mathcal{L} = (D_\mu\phi)^* D^\mu\phi + \frac{\kappa}{4\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda \tag{1}$$

where $D_\mu = \partial_\mu - \mathrm{i}A_\mu$ and $\epsilon_{012} = 1$. It is invariant under the $U(1)$ gauge transformations

$$\phi(x) \to \mathrm{e}^{\mathrm{i}\Lambda(x)}\phi(x) \qquad A_\mu(x) \to A_\mu(x) + \partial_\mu\Lambda(x) \tag{2}$$

where it is assumed that the transformation function Λ has vanishing boundary condition at infinity; $\Lambda(\infty) = 0$.

The remaining sections of this paper are composed as follows. In section 2, the system (1) is quantized in the Coulomb gauge. One type of gauge-invariant composite operator is considered as the candidate for the anyon operator in section 3. We investigate the anyonicity of it and prove the generalized spin-statistics relation in section 4. Finally, in section 5, we show that the anyonicity obtained in section 4 is also valid in the Lorentz covariant gauge, and give conclusions.

2. Quantization

We quantize the system (1) in the Hamiltonian formulation. Because the system (1) possesses constraints, we will adopt the Dirac quantization procedure [9] to treat them and work out in the Coulomb gauge.

The Hamiltonian density corresponding to the system (1) is

$$\mathcal{H} = \Pi\,\Pi^* - (D_i\phi)^* D^i\phi - A_0\left(\frac{\kappa}{2\pi}\epsilon^{ij}\partial_i A_j - J_0\right) \tag{3}$$

where Π (Π^*) is the canonical momentum conjugate to the ϕ (ϕ^*) field, $J_0 = \mathrm{i}(\Pi\phi - \phi^*\Pi^*)$ is the charge density of the matter field, and $\epsilon^{ij} \equiv \epsilon^{0ij}$. The canonical momenta conjugate to A_0 and A_i are $\Pi_0 \approx 0$ and $\Pi_i = (\kappa/4\pi)\epsilon_{ij}A^j$, respectively. The primary constraints are

$$\Gamma_0 = \Pi_0 \approx 0 \qquad \Gamma_i = \Pi_i - \frac{\kappa}{4\pi}\epsilon_{ij}A^j \approx 0 \quad (i, j = 1, 2) \tag{4}$$

and the secondary constraint is

$$\Gamma_3 = \frac{\kappa}{2\pi}\epsilon^{ij}\partial_i A_j - J_0 \approx 0 \tag{5}$$

which is just the Gauss-law constraint. After fixing the gauge (Coulomb gauge) we obtain the additional constraints involving the gauge condition as follows:

$$\Gamma_4 = \partial_i A^i \approx 0 \tag{6}$$

$$\Gamma_5 = -\nabla^2 A_0 + \frac{2\pi}{\kappa}\epsilon_{ij}\partial^i J^j \approx 0 \tag{7}$$

where $\nabla^2 = -\partial_i\partial^i$ and $J^j = \mathrm{i}(D^j\phi)^*\phi - \mathrm{i}\phi^*(D^j\phi)$. These conditions and those of (4) and (5) form a fully second class, and therefore one is now ready to introduce the Dirac brackets, $\{\,,\}_D$, and obtain the equal-time commutators through the replacement of $\{\,,\}_D$ by $-\mathrm{i}[\,,]$. Here it should be noted that we treat Γ_0 and Γ_5 independently from the other constraints and solve them directly. This is a safe treatment since the constraint themselves except Γ_0 and Γ_5 form a fully second class. Then the non-vanishing equal-time commutators are as follows:

$$\begin{aligned}
&[\phi(\boldsymbol{x}), \Pi(\boldsymbol{y})] = \mathrm{i}\delta(\boldsymbol{x}-\boldsymbol{y})\\
&\left[\phi^*(\boldsymbol{x}), \Pi^*(\boldsymbol{y})\right] = \mathrm{i}\delta(\boldsymbol{x}-\boldsymbol{y})\\
&[\phi(\boldsymbol{x}), A_i(\boldsymbol{y})] = -\frac{2\pi}{\kappa}\phi(\boldsymbol{x})\epsilon_{ij}\partial_x^j G(\boldsymbol{x}-\boldsymbol{y})\\
&\left[\phi^*(\boldsymbol{x}), A_i(\boldsymbol{y})\right] = \frac{2\pi}{\kappa}\phi^*(\boldsymbol{x})\epsilon_{ij}\partial_x^j G(\boldsymbol{x}-\boldsymbol{y})\\
&[\Pi(\boldsymbol{x}), A_i(\boldsymbol{y})] = \frac{2\pi}{\kappa}\Pi(\boldsymbol{x})\epsilon_{ij}\partial_x^j G(\boldsymbol{x}-\boldsymbol{y})\\
&\left[\Pi^*(\boldsymbol{x}), A_i(\boldsymbol{y})\right] = -\frac{2\pi}{\kappa}\Pi^*(\boldsymbol{x})\epsilon_{ij}\partial_x^j G(\boldsymbol{x}-\boldsymbol{y})
\end{aligned} \tag{8}$$

where $G(\boldsymbol{x}-\boldsymbol{y})$ is the Green function in two spatial dimensions which satisfies $\nabla^2 G(\boldsymbol{x}) = -\delta(\boldsymbol{x})$ and has the form $G(\boldsymbol{x}) = -(1/2\pi)\ln\mu|\boldsymbol{x}|$, where μ is the infrared cutoff. Here it should be noted that $\epsilon_{ij}\partial^j G$ in the above commutation relations is ill defined at the origin. To resolve this ill-definedness, we follow the prescription of Jackiw and Pi [10] that $\epsilon_{ij}\partial^j G$ vanishes at the origin by taking a certain regularization which preserves the anti-symmetric property of it under the space reflection.

3. Composite operators

For the construction of the gauge-invariant anyon operators, we consider the composite operator which is defined by

$$\hat{\phi}(x) = \phi(x)\,\exp\left(-\mathrm{i}\int \mathrm{d}^2y\;D^i(\boldsymbol{x}-\boldsymbol{y})\,A_i(x_0,\boldsymbol{y})\right) \tag{9}$$

where D^i is the distribution function which makes $\hat{\phi}$ gauge invariant and not yet determined. Requiring the invariance of $\hat{\phi}$ under the gauge transformation (2), we obtain the equation that D^i must satisfy as follows:

$$\partial_i^y D^i(\boldsymbol{x}-\boldsymbol{y}) = -\delta(\boldsymbol{x}-\boldsymbol{y})\,. \tag{10}$$

We note that, in the covariant gauge, this equation also ensures the invariance of $\hat{\phi}(x)$ under the BRST transformation [11].

In order to obtain the solution of (10) which is appropriate to the purpose of this paper, we now follow the solution given in [7, 11] and briefly review it in a self-contained manner. In momentum space, the parametrized solution of (10) is

$$D^i(p) = \mathrm{i} \int_0^{2\pi} \mathrm{d}\varphi\, \tau(\varphi) \frac{n^i}{n \cdot p + \mathrm{i}\epsilon} \tag{11}$$

where φ is a parameter which parametrizes the solutions, $n = (\cos\varphi, \sin\varphi)$, and $\tau(\varphi)$ is a density function which satisfies $\int_0^{2\pi} \mathrm{d}\varphi\, \tau(\varphi) = 1$. Then the Fourier transformation of (11) leads to the solution in configuration space as follows:

$$D^i(x) = \int_0^\infty \mathrm{d}s\, \mathrm{e}^{-s\epsilon} \int_0^{2\pi} \mathrm{d}\varphi\, \tau(\varphi) n^i \delta(x - sn)\,. \tag{12}$$

The support properties of D^i are determined by the density τ. If we let the support of τ be

$$\mathrm{supp}\, \tau(\varphi) = [\alpha_1, \alpha_2] \qquad 0 < \alpha_2 - \alpha_1 < \pi$$

then the support of D^i is given by the wedge as follows:

$$W = \{(|x| \cos\varphi, |x| \sin\varphi) \in \mathbb{R}^2 : \alpha_1 \leqslant \varphi \leqslant \alpha_2\}\,.$$

Now we are interested in the case in which the wedge W degenerates into a space-like string S. In this case, the distribution function D^i may be represented by using any weakly convergent sequence of densities $\{\tau_a(\varphi) : a = 1, 2, \ldots\}$ such that

$$\lim_{a\to\infty} \tau_a(\varphi) = \delta(\varphi - \alpha) \qquad 0 \leqslant \alpha < 2\pi \tag{13}$$

where α specifies the direction of the string S. Here we take $\alpha = \pi$ for convenience. Let us perform the integration in (12) by using (13) and $\alpha = \pi$ to know what form the distribution D^i takes. Then we obtain

$$D^i(x) = -\delta^{i1} \theta(-x^1) \delta(x^2) \tag{14}$$

where θ is the step function. Apparently, this satisfies (10) and describes the infinitely long space-like string sitting at the negative x^1-axis. Returning to the definition of $\hat{\phi}$ (9), one finds that this makes the operator $\hat{\phi}$ into an object localized in a space-like string.

4. Anyonicity and the spin-statistics relation

To study the anyonicity of composite operators and the spin-statistics relation, we first obtain the charge and magnetic flux carried by the operator (9). By definition, the charge and magnetic flux operators are given by

$$Q = \int \mathrm{d}^2x\, J_0 \tag{15}$$

$$\Phi = \int \mathrm{d}^2x\, \epsilon^{ij} \partial_i A_j \tag{16}$$

respectively. Using the commutators (8), we derive

$$[\, Q, \hat{\phi}(x)\,] = \hat{\phi}(x) \tag{17}$$

$$[\, \Phi, \hat{\phi}(x)\,] = \frac{2\pi}{\kappa} \hat{\phi}(x) \tag{18}$$

which imply that the operator $\hat{\phi}$ carries one unit of positive charge and magnetic flux of amount $2\pi/\kappa$. Now we regard $\hat{\phi}$ as the creation operator of one antiparticle, and one (anti) particle state as carrying one unit of (positive) negative charge following [12]. Then $\hat{\phi}$ is the operator which creates the physical one-antiparticle state carrying charge and magnetic flux from the vacuum, and shows some aspect of the prototypical quantum mechanical anyon by Wilczek [1].

The statistics obeyed by the operator $\hat{\phi}$ may be measured by considering the process of exchanging two operators in the product $\hat{\phi}(x)\hat{\phi}(y)$. By using the definition (9) of $\hat{\phi}$ and the commutators (8), we obtain

$$\hat{\phi}(x)\hat{\phi}(y) = \exp\left(\mathrm{i}\frac{2\pi}{\kappa}\int \mathrm{d}^2z\, D^i(x-z)\epsilon_{ij}\partial_z^j G(z-y)\right) \times \exp\left(-\mathrm{i}(2\pi/\kappa)\int \mathrm{d}^2z\, D^i(y-z)\epsilon_{ij}\partial_z^j G(z-x)\right)\hat{\phi}(y)\hat{\phi}(x)\,. \tag{19}$$

The integration in the exponential factor of (19) is easily performed by substituting the solution (12) for D^i and taking (13) at the final step, and the result of the integration is

$$\int \mathrm{d}^2z\, D^i(x-z)\epsilon_{ij}\partial_z^j G(z-y) = \frac{1}{2\pi}\tan^{-1}\frac{y^2-x^2}{y^1-x^1} \tag{20}$$

which is just the angle function. Now we define

$$\Theta(x-y) = \frac{1}{2\pi}\tan^{-1}\frac{x^2-y^2}{x^1-y^1}\,. \tag{21}$$

Then equation (19) becomes

$$\begin{aligned}\hat{\phi}(x)\hat{\phi}(y) &= \mathrm{e}^{\mathrm{i}(2\pi/\kappa)[\Theta(y-x)-\Theta(x-y)]}\hat{\phi}(y)\hat{\phi}(x)\\ &= \mathrm{e}^{\pm\mathrm{i}\pi/\kappa}\hat{\phi}(y)\hat{\phi}(x)\end{aligned} \tag{22}$$

where the following relation is used:

$$\Theta(y-x) - \Theta(x-y) = \pm\tfrac{1}{2}\,. \tag{23}$$

The sign ambiguity in (22) stems from the fact that the function $\tan^{-1}$ is defined only in modulo 2π. In a physical sense, this corresponds to the two possible ways of rotation of one particle around the other, namely clockwise and anticlockwise rotation. In the former (latter) case the sign in (22) is minus (plus).

We are now in a position to discuss the statistics of the operator $\hat{\phi}$. From the graded commutation relation (22), it follows that the CS coefficient κ determines the statistics of $\hat{\phi}$ and thus the operator $\hat{\phi}$ may obey arbitrary statistics since there are no restrictions to the value of κ. As particular cases, if κ is taken as

$$\kappa = \frac{1}{2n+1} \qquad (n \in \mathbb{Z}) \tag{24}$$

then equation (22) becomes the anticommutation relation and hence the operator $\hat{\phi}$ obeys Fermi statistics. If κ is taken as

$$\kappa = \frac{1}{2n} \qquad (n \in \mathbb{Z}) \tag{25}$$

the operator $\hat{\phi}$ obeys Bose statistics.

The graded commutation relations such as (22) may be obtained for the other gauge-invariant composite operators defined by

$$\hat{\phi}^*(x) = \exp\left(\mathrm{i}\int \mathrm{d}^2y\, D^i(\boldsymbol{x}-\boldsymbol{y})A_i(x_0,\boldsymbol{y})\right)\phi^*(x) \tag{26}$$

$$\hat{\Pi}(x) = \exp\left(\mathrm{i}\int \mathrm{d}^2y\, D^i(\boldsymbol{x}-\boldsymbol{y})A_i(x_0,\boldsymbol{y})\right)\Pi(x) \tag{27}$$

$$\hat{\Pi}^*(x) = \Pi^*(x)\exp\left(-\mathrm{i}\int \mathrm{d}^2y\, D^i(\boldsymbol{x}-\boldsymbol{y})A_i(x_0,\boldsymbol{y})\right). \tag{28}$$

Through the same procedure up to (22), we obtain the graded commutation relations between operators $\hat{\phi}$, $\hat{\phi}^*$, $\hat{\Pi}$, and $\hat{\Pi}^*$ as follows:

$$\begin{aligned}
&\hat{\phi}(\boldsymbol{x})\hat{\phi}(\boldsymbol{y}) - \mathrm{e}^{\pm\mathrm{i}\pi/\kappa}\hat{\phi}(\boldsymbol{y})\hat{\phi}(\boldsymbol{x}) = 0\\
&\hat{\phi}(\boldsymbol{x})\hat{\phi}^*(\boldsymbol{y}) - \mathrm{e}^{\mp\mathrm{i}\pi/\kappa}\hat{\phi}^*(\boldsymbol{y})\hat{\phi}(\boldsymbol{x}) = 0\\
&\hat{\phi}(\boldsymbol{x})\hat{\Pi}^*(\boldsymbol{y}) - \mathrm{e}^{\pm\mathrm{i}\pi/\kappa}\hat{\Pi}^*(\boldsymbol{y})\hat{\phi}(\boldsymbol{x}) = 0\\
&\hat{\Pi}(\boldsymbol{x})\hat{\Pi}^*(\boldsymbol{y}) - \mathrm{e}^{\mp\mathrm{i}\pi/\kappa}\hat{\Pi}^*(\boldsymbol{y})\hat{\Pi}(\boldsymbol{x}) = 0\\
&\hat{\phi}(\boldsymbol{x})\hat{\Pi}(\boldsymbol{y}) - \mathrm{e}^{\mp\mathrm{i}\pi/\kappa}\hat{\Pi}(\boldsymbol{y})\hat{\phi}(\boldsymbol{x}) = \mathrm{i}\delta(\boldsymbol{x}-\boldsymbol{y})
\end{aligned} \tag{29}$$

where we have rewritten (22) for summary.

It is not obvious whether or not the statistics interpretation given above is correct and consistent. In order to clarify this point, we first consider the definition of statistics by looking at the state functional. The general form of the N-particle state functional following from the representation theory of the braid group is given by

$$\begin{aligned}
&\Psi_\sigma\left[\phi^*(\boldsymbol{x}_1),\ldots,\phi^*(\boldsymbol{x}_N);t\right]\\
&\quad= \exp\left[\mathrm{i}2\sigma\sum_{j=1}^{N}\sum_{i=1}^{j-1}2\pi\Theta(\boldsymbol{x}_i-\boldsymbol{x}_j)\right]\Psi_0\left[\phi^*(\boldsymbol{x}_1),\ldots,\phi^*(\boldsymbol{x}_N);t\right]
\end{aligned} \tag{30}$$

where $\Psi_0[\phi^*(\boldsymbol{x}_1),\ldots,\phi^*(\boldsymbol{x}_N);t]$ is an N-particle state functional with Bose statistics and σ is, by definition, the statistics [12]. For the integer values of σ, the state functional (30) describes the N particles with Bose statistics. Also, for the half-integer values of σ, (30) describes the N particles with Fermi statistics. Now we construct the N-particle state by acting the operator $\hat{\phi}^*$ N times on the vacuum and comparing it with the state functional (30). The N-particle state $|N\rangle$ is constructed as

$$|N\rangle = \prod_{i=1}^{N}\hat{\phi}^*(\boldsymbol{x}_i)|0\rangle = \prod_{i=1}^{N}\exp\left(\mathrm{i}\int\mathrm{d}^2y\, D^j(\boldsymbol{x}_i-\boldsymbol{y})A_j(\boldsymbol{y})\right)\phi^*(\boldsymbol{x}_i)|0\rangle. \tag{31}$$

In order to relate the state $|N\rangle$ to (30), we change the form of $|N\rangle$ to that of (30). Here we note that the following Baker–Campbell–Hausdorff formula is useful:

$$\begin{aligned}
&\exp\left(-\mathrm{i}\int\mathrm{d}^2z\, D^i(\boldsymbol{y}-\boldsymbol{z})A_i(\boldsymbol{z})\right)\phi^*(\boldsymbol{x})\exp\left(\mathrm{i}\int\mathrm{d}^2z\, D^i(\boldsymbol{y}-\boldsymbol{z})A_i(\boldsymbol{z})\right)\\
&\quad= \mathrm{e}^{-\mathrm{i}(2\pi/\kappa)\Theta(\boldsymbol{x}-\boldsymbol{y})}\phi^*(\boldsymbol{x})
\end{aligned} \tag{32}$$

where the commutators (8) and (20) are used. Using this formula, we rewrite the N-particle state $|N\rangle$ as

$$|N\rangle = \exp\left[-\mathrm{i}\frac{1}{\kappa}\sum_{j=1}^{N}\sum_{i=1}^{j-1}2\pi\Theta(\boldsymbol{x}_i-\boldsymbol{x}_j)\right] \times \left\{\exp\left(\mathrm{i}\sum_{i=1}^{N}\int \mathrm{d}^2y\, D^j(\boldsymbol{x}_i-\boldsymbol{y})A_j(\boldsymbol{y})\right)\prod_{i=1}^{N}\phi^*(\boldsymbol{x}_i)|0\rangle\right\}. \tag{33}$$

The expression enclosed in the curly braces is gauge invariant and independent of interchanges of any number of coordinates. Thus it is an N-particle state with Bose statistics and hence may correspond to Ψ_0 of (30). For the remaining expression containing the function Θ, the correspondence between (30) and (33) gives a relation naturally between the statistics σ and the CS coefficient κ as follows:

$$\sigma = -\frac{1}{2\kappa}. \tag{34}$$

With this relation, we may now check whether the statistics interpretation given by the values of κ, in particular by (24) and (25), is correct. For the case of Fermi statistics, σ takes half-integer values and these just lead to the values of κ in (24) for which our composite operators satisfy the anticommutation relations. While, for the case of Bose statistics, σ takes integer values and these lead to (25). Therefore it becomes clear that the statistics interpretation given by the values of κ is correct.

The N-particle state (33) is multi-valued due to the presence of function Θ. Introducing complex coordinates $z_i = x_i^1 + \mathrm{i}x_i^2$, the multi-valuedness of $|N\rangle$ may become more transparent as follows:

$$|N\rangle = \prod_{i<j}(z_i - z_j)^{-1/\kappa}|\,\text{single-valued}\,\rangle. \tag{35}$$

This state is similar to that constructed from Laughlin's ansatz [13].

So far, it has been shown that the gauge-invariant composite operator considered here obeys the generalized statistics and leads to the multi-valued, N-particle states which have the same form as those following from the representation theory of the braid group. Therefore, it may be concluded that the operator $\hat{\phi}$ is the anyon field operator.

In connection with the generalized spin-statistics relation, we now compute the spin s of one-particle (anyon) state $\hat{\phi}^*|0\rangle$ and connect the result with relation (34). For this purpose, we first construct the angular momentum operator L which is defined by

$$L = \int \mathrm{d}^2x\, \epsilon^{ij}x_i T_{0j} \tag{36}$$

where T_{0j} is an element of the symmetric energy–momentum tensor. The symmetric energy–momentum tensor for the system (1) is derived from

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int \mathrm{d}^3x\, \mathcal{L}$$
$$= (D_\mu\phi)^* D_\nu\phi + (D_\nu\phi)^* D_\mu\phi - g_{\mu\nu}(D_\alpha\phi)^* D^\alpha\phi. \tag{37}$$

Hence

$$L = \int \mathrm{d}^2x\, \epsilon^{ij}x_i(\Pi\partial_j\phi + \partial_j\phi^*\Pi^*) - \int \mathrm{d}^2x\, \epsilon^{ij}x_i A_j J_0. \tag{38}$$

Since the solution of the Gauss-law Γ_3 is given by

$$A_i(x) = -\frac{2\pi}{\kappa}\epsilon_{ij}\partial_x^j \int d^2y\, G(x-y)J_0(y) \tag{39}$$

after substituting this solution into (38) and performing some manipulations following the usual manner [14], we finally obtain

$$L = \int d^2x\, \epsilon^{ij}x_i(\Pi\partial_j\phi + \partial_j\phi^*\Pi^*) + \frac{1}{2\kappa}Q^2 \tag{40}$$

where the first term on the right-hand side is the canonical angular momentum operator and the second is the anomalous one which is interpreted as a spin operator [14]. Since these two terms commute with each other we may treat them independently, and concentrate only on the spin operator in order to obtain the spin of the one-particle state.

Now we denote the spin operator by S: $S = \frac{1}{2\kappa}Q^2$. Then, if we rotate the one-particle state with S, we obtain

$$e^{i\theta S}\hat{\phi}^*|0\rangle = e^{i\theta(1/2\kappa)}\hat{\phi}^*|0\rangle \tag{41}$$

where θ is the rotation parameter. The eigenvalue of the spin operator S is the spin s, and thus we obtain a relation between the spin s and the CS coefficient κ, namely

$$s = \frac{1}{2\kappa}. \tag{42}$$

We now take θ as 2π. For $\kappa = 1/(2n+1)$ $(n \in \mathbb{Z})$, the one-particle state picks up a minus sign implying that it is fermionic, and these values of κ let the spin s take half-integer values as seen from the relation (42). While, for $\kappa = 1/2n$ $(n \in \mathbb{Z})$, the one-particle state does not change and hence it becomes bosonic, and the spin s takes integer values. For the other values of κ, the state becomes anyonic and the spin s is fractional.

All the arguments given here are in complete agreement with those for the statistics interpretation, in particular for the Fermi (24) and Bose (25) statistics. Thus we now have the consistent and correct interpretation for the spin and statistics starting from the gauge-invariant anyon operator $\hat{\phi}$. Furthermore, by comparing the relations (34) and (42), we can obtain a new relation as follows:

$$\sigma = -s. \tag{43}$$

This establishes a generalized spin-statistics relation and is the same as that concluded in a work by Forte and Jolicœur [12], in which this relation was obtained by studying the symmetry structure of the relativistic field theory whose Fock states provide a multi-valued representation of the Poincaré group.

5. Discussion and conclusion

In the Hamiltonian framework, we have shown that the gauge-invariant composite operator $\hat{\phi}$ satisfies the anyonic properties, and proved the generalized spin-statistics relation. However, all the formulations have been done in the Coulomb gauge. As was suggested in section 1, if the anyonicity of $\hat{\phi}$ obtained in that gauge is really gauge-independent and thus physically meaningful, it should also be obtained in the other gauges. The anyonicity originates from the graded commutation relation. Now, in order to show that it is not a gauge artifact, we take the Lorentz covariant gauge and compute the product $\hat{\phi}(x)\hat{\phi}(y)$.

In the Lorentz covariant gauges, the only non-trivial commutators are

$$[A_i(x), A_j(y)] = i\frac{2\pi}{\kappa}\epsilon_{ij}\delta(x-y) \tag{44}$$

and the other commutators are canonical [8]. Then the product $\hat{\phi}(x)\hat{\phi}(y)$ is evaluated as

$$\hat{\phi}(x)\hat{\phi}(y) = e^{i(\pi/\kappa)\Delta(x-y)}\hat{\phi}(y)\hat{\phi}(x) \tag{45}$$

where

$$\Delta(x-y) = -2\epsilon_{ij}\int d^2z\, D^i(x-z)D^j(y-y)\,. \tag{46}$$

Following the analysis in [7, 11] for the case in which the support of the distribution function D^i degenerates into a space-like string, $\Delta(x-y)$ reads

$$\Delta(x-y) = \begin{cases} +1 & \text{anticlockwise} \\ -1 & \text{clockwise} \end{cases} \tag{47}$$

where (anti) clockwise corresponds to the situation that y is moved around x in a (anti) clockwise sense. Thus equation (45) becomes

$$\hat{\phi}(x)\hat{\phi}(y) - e^{\pm i\pi/\kappa}\hat{\phi}(y)\hat{\phi}(x) = 0 \tag{48}$$

which is just the graded commutation relation (22) derived in the Coulomb gauge. Since this commutator implies the anyonicity of $\hat{\phi}$, it may be concluded that the anyonicity is still valid in the Lorentz covariant gauge.

The gauge independence of the anyonicity puts the identification of the gauge-invariant operator $\hat{\phi}$ with the anyon field on a firmer footing, and means that the composite operator $\hat{\phi}$ may become the fundamental field of the anyon field theory.

We now return to the anyonicity itself and make some comments on the role of its origin. Reviewing our formulations, the anyonicity of $\hat{\phi}$ comes from the two independent factors which are the presence of the CS term and the distribution function D^i. The graded commutators which are the relations between multianyons come from the cooperations of these two factors. On the other hand, the appearance of both charge and magnetic flux, which is the property of the single anyon, is governed only by the CS term. In this sense, the role of the CS term is two-fold, while that of the distribution function D^i is single-fold.

In conclusion, we have studied the gauge-invariant anyon operator $\hat{\phi}$ which is constructed based on relativistic field theory and we have shown that the anyonic properties exist. The generalized spin-statistics relation has been obtained.

Finally, some comments are in order. Firstly, the operator studied so far does not contain the angle function, which appears only in the final result as, for example, in equation (20). Thus it may be different from that which can be seen in some previous works [4, 14] where the angle function was contained explicitly in the anyon operator. Secondly, the system considered here is the complex scalar field theory coupled to the CS term, which may be thought of as the simplest one allowing anyonic excitations. As a further study in this direction, one may consider the Maxwell–Chern–Simons theory with relativistic matter in order to investigate whether the properties of anyons still survive after inclusion of the Maxwell kinetic term.

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